The Pythagorean Theorem and Its Converse

What You’ll Learn

• To use the Pythagorean Theorem
• To use the Converse of the Pythagorean Theorem

...And Why

To find the distance between two docks on a lake, as in Example 3

The Pythagorean Theorem

The well-known right triangle relationship called the Pythagorean Theorem is named for Pythagoras, a Greek mathematician who lived in the sixth century B.C. We now know that the Babylonians, Egyptians, and Chinese were aware of this relationship before its discovery by Pythagoras.

There are many proofs of the Pythagorean Theorem. You will see one proof in Exercise 48 and others later in the book.

A Pythagorean triple is a set of nonzero whole numbers $a$, $b$, and $c$ that satisfy the equation $a^2 + b^2 = c^2$. Here are some common Pythagorean triples.

$3, 4, 5$  
$5, 12, 13$  
$8, 15, 17$  
$7, 24, 25$

If you multiply each number in a Pythagorean triple by the same whole number, the three numbers that result also form a Pythagorean triple.

New Vocabulary  
Pythagorean triple

Check Skills You’ll Need

Square the lengths of the sides of each triangle. What do you notice?

1. $3^2 + 4^2 = 5^2$
2. $5^2 + 12^2 = 13^2$
3. $6^2 + 8^2 = 10^2$
4. $4^2 + 2^2 = (4\sqrt{2})^2$

Math Background

Some mathematical ideas assumed to be true have yet to be proved, such as Goldbach’s conjecture: Every even number greater than 2 can be expressed as the sum of two prime numbers. Although several ancient cultures postulated the Pythagorean Theorem and used it to measure distances, the first proof of it was attributed by Euclid to Pythagoras. The distance formula is a coordinate form of the Pythagorean Theorem, which is the foundation of all trigonometric functions.

Lesson Planning and Resources

See p. 414E for a list of the resources that support this lesson.

Bell Ringer Practice

For intervention, direct students to: Skills Handbook, p. 753

Check Skills You’ll Need

Before the lesson, list the squares of whole numbers less than 20. Also review how to simplify a radical expression.
2. **Teach**

**Guided Instruction**

1. **EXAMPLE** Teaching Tip
   
   Memorizing the common Pythagorean triples, like those at the bottom of p. 417, can help you solve problems more quickly.

2. **EXAMPLE** Error Prevention
   
   Some students may assume that the legs are always the known quantities. Point out that c is always the hypotenuse when applying the formula \( a^2 + b^2 = c^2 \) to a right triangle.

3. **EXAMPLE** Technology Tip
   
   Students may wonder why they are asked to use a calculator in some exercises but not in other similar exercises. Tell them that real-world applications typically require decimal answers. Point out that radicals are exact, so they are preferred when exercises are of a purely mathematical nature.

**Additional Examples**

1. A right triangle has legs of length 16 and 30. Find the length of the hypotenuse. Do the lengths of the sides form a Pythagorean triple? 34; yes
2. Find the value of \( x \). Leave your answer in simplest radical form.
3. A baseball diamond is a square with 90-ft sides. Home plate and second base are at opposite vertices of the square. About how far is home plate from second base? about 127 ft

**Test-Taking Tip**

Memorizing the common Pythagorean triples, like those at the bottom of p. 417, can help you solve problems more quickly.

**EXAMPLE** Pythagorean Triples

Find the length of the hypotenuse of \( \triangle ABC \). Do the lengths of the sides of \( \triangle ABC \) form a Pythagorean triple?

\[
\begin{align*}
21^2 + 20^2 &= c^2 \\
441 + 400 &= c^2 \\
841 &= c^2 \\
c &= 29
\end{align*}
\]

The length of the hypotenuse is 29. The lengths of the sides, 20, 21, and 29, form a Pythagorean triple because they are whole numbers that satisfy \( a^2 + b^2 = c^2 \).

**Quick Check**

1. A right triangle has a hypotenuse of length 25 and a leg of length 10. Find the length of the other leg. Leave your answer in simplest radical form.
2. The hypotenuse of a right triangle has length 12. One leg has length 6. Find the length of the other leg. Leave your answer in simplest radical form.
3. The hypotenuse of a right triangle has length 12. One leg has length 6. Find the length of the other leg. Leave your answer in simplest radical form.

**EXAMPLE** Using Simplest Radical Form

**Algebra** Find the value of \( x \). Leave your answer in simplest radical form (page 390).

\[
\begin{align*}
8^2 + x^2 &= 20^2 \\
64 + x^2 &= 400 \\
x^2 &= 336 \\
x &= \sqrt{336} \\
x &= 4\sqrt{21}
\end{align*}
\]

2. **EXAMPLE** Using Simplest Radical Form

**Gridded Response** The Parks Department rents paddle boats at docks near each entrance to the park. To the nearest meter, how far is it to paddle from one dock to the other?

\[
\begin{align*}
250^2 + 350^2 &= c^2 \\
625,000 &= c^2 \\
c &= \sqrt{625,000} \\
c &= 2,500 \\
c &= 721.11626
\end{align*}
\]

It is 721 m from one dock to the other.

**Quick Check**

1. A right triangle has legs of length 16 and 30. Find the length of the hypotenuse. Do the lengths of the sides form a Pythagorean triple? 34; yes
2. Find the value of \( x \). Leave your answer in simplest radical form.
3. A baseball diamond is a square with 90-ft sides. Home plate and second base are at opposite vertices of the square. About how far is home plate from second base? about 127 ft

**Critical Thinking** When you want to know how far you have to paddle a boat, why is an approximate answer more useful than an answer in simplest radical form? You want to know the nearest whole number value, which may not be apparent in a radical expression.

**Differentiated Instruction**

- **Advanced Learners**
  
  Have students describe how a triangle whose sides form a Pythagorean triple and a triangle whose sides are a multiple of that triple are related. Students should recognize that they are similar triangles.

- **English Language Learners**
  
  Review the term converse, using the Pythagorean Theorem and its converse as an example. Then have students write the Pythagorean Theorem as a biconditional statement.

**Solutions for All Learners**

Learning style: verbal
The Converse of the Pythagorean Theorem

You can use the Converse of the Pythagorean Theorem to determine whether a triangle is a right triangle. You will prove Theorem 8-2 in Exercise 58.

**Theorem 8-2**  
Converse of the Pythagorean Theorem  
If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

**Example**  
Is this triangle a right triangle?

\[ c^2 = a^2 + b^2 \]

\[ 85^2 = 84^2 + 13^2 \]

\[ 7225 = 7056 + 169 \]

\[ 7225 = 7225 \]

✓  

c^2 = a^2 + b^2, so the triangle is a right triangle.

A triangle has sides of lengths 16, 48, and 50. Is the triangle a right triangle?

You can also use the squares of the lengths of the sides of a triangle to find whether the triangle is acute or obtuse. The following two theorems tell how.

**Theorem 8-3**  
If the square of the length of the longest side of a triangle is greater than the sum of the squares of the lengths of the other two sides, the triangle is obtuse.  
If \( c^2 > a^2 + b^2 \), the triangle is obtuse.

**Theorem 8-4**  
If the square of the length of the longest side of a triangle is less than the sum of the squares of the lengths of the other two sides, the triangle is acute.  
If \( c^2 < a^2 + b^2 \), the triangle is acute.

**Example**  
Classifying Triangles as Acute, Obtuse, or Right  
Classify the triangle whose side lengths are 6, 11, and 14 as acute, obtuse, or right.  
\[ 14^2 = 6^2 + 11^2 \]  
\[ 196 = 36 + 121 \]

\[ 196 > 157 \]

Since \( c^2 > a^2 + b^2 \), the triangle is obtuse.

A triangle has sides of lengths 7, 8, and 9. Classify the triangle by its angles.

**Quick Check**

A triangle has sides of lengths 16, 48, and 50. Is the triangle a right triangle?

You can also use the squares of the lengths of the sides of a triangle to find whether the triangle is acute or obtuse. The following two theorems tell how.

**Theorem 8-2**  
Converse of the Pythagorean Theorem  
If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

**Example**  
Is this triangle a right triangle?

\[ c^2 = a^2 + b^2 \]

\[ 85^2 = 6^2 + 11^2 \]  
\[ 85^2 = 36 + 121 \]

\[ 85^2 = 157 \]

✓  

c^2 = a^2 + b^2, so the triangle is a right triangle.

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**Example**  
Classifying Triangles as Acute, Obtuse, or Right  
Classify the triangle whose side lengths are 6, 11, and 14 as acute, obtuse, or right.  
\[ 14^2 = 6^2 + 11^2 \]  
\[ 196 = 36 + 121 \]

\[ 196 > 157 \]

Since \( c^2 > a^2 + b^2 \), the triangle is obtuse.

A triangle has sides of lengths 7, 8, and 9. Classify the triangle by its angles.

**Quick Check**

A triangle has sides of lengths 16, 48, and 50. Is the triangle a right triangle?
Find the value of $x$.

1. $8 \div x = 10$
2. $25 \div x = 7$
3. $x \div 16 = 34$
4. $20 \div x = 12$
5. $65 \div x = 97$
6. $8 \div x = 15$

Does each set of numbers form a Pythagorean triple? Explain.

7. 4, 5, 6
   - no; $4^2 + 5^2 \neq 6^2$
8. 10, 24, 26
   - yes; $10^2 + 24^2 = 26^2$
9. 15, 20, 25
   - yes; $15^2 + 20^2 = 25^2$

Example 3

16. **Home Maintenance** A painter leans a 15-ft ladder against a house. The base of the ladder is 5 ft from the house.
   - **a.** To the nearest tenth of a foot, how high on the house does the ladder reach? **14.1 ft**
   - **b.** The ladder in part (a) reaches too high on the house. By how much should the painter move the ladder’s base away from the house to lower the top by 1 ft? **about 2.3 ft**

17. A walkway forms the diagonal of a square playground. The walkway is 24 m long. To the nearest tenth of a meter, how long is a side of the playground? **17.0 m**

Example 4

20. Is each triangle a right triangle? Explain.
   - **19.**
     - no; $8^2 + 24^2 \neq 25^2$
   - **20.**
     - yes; $33^2 + 56^2 = 65^2$

Example 5

The lengths of the sides of a triangle are given. Classify each triangle as acute, right, or obtuse.

- **21.** 4, 5, 6 **acute**
- **22.** 0.3, 0.4, 0.6 **obtuse**
- **23.** 11, 12, 15 **acute**
- **24.** $\sqrt{3}, 2, 3$ **obtuse**
- **25.** 30, 40, 50 **right**
- **26.** $\sqrt{7}, \sqrt{7}, 4$ **acute**
30. **Writing** Each year in an ancient land, a large river overflowed its banks, often destroying boundary markers. The royal surveyors used a rope with knots at 12 equal intervals to help reconstruct boundaries. Explain how a surveyor could use this rope to form a right angle. (Hint: Use the Pythagorean triple 3, 4, 5.)

31. **Multiple Choice** Which triangle is not a right triangle?  

   - A. ![Triangle A](image)
   - B. ![Triangle B](image)
   - C. ![Triangle C](image)
   - D. ![Triangle D](image)

   **Answer:** B

32. **Embroidery** You want to embroider a square design. You have an embroidery hoop with a 6 in. diameter. Find the largest value of \( x \) so that the entire square will fit in the hoop. Round to the nearest tenth. **4.2 in.**

33. In parallelogram \( RSTW \), \( RS = 7, ST = 24 \), and \( RT = 25 \). Is \( RSTW \) a rectangle? Explain. **Yes; 7^2 + 24^2 = 25^2, so \( \triangle RST \) is a rt. \( \angle \).**

34. **Coordinate Geometry** You can use the Pythagorean Theorem to prove the Distance Formula. Let points \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) be the endpoints of the hypotenuse of a right triangle.

   a. Write an algebraic expression to complete each of the following:
   
   \[ PR = \quad \text{and} \quad QR = \quad |x_2 - x_1| \quad |y_2 - y_1| \]

   b. By the Pythagorean Theorem,
   
   \[ PQ^2 = PR^2 + QR^2 \]
   
   Rewrite this statement substituting the algebraic expressions you found for \( PR \) and \( QR \) in part (a).

   c. Complete the proof by taking the square root of each side of the equation that you wrote in part (b).
   
   \[ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

35. **Constructions** Explain how to construct a segment of length \( \sqrt{n} \), where \( n \) is any positive integer, and you are given a segment of length 1. (Hint: See the diagram.) See margin.

   Find a third whole number so that the three numbers form a Pythagorean triple.
   
   - 20, 21 **29**
   - 14, 48 **50**
   - 13, 85 **84**
   - 12, 37 **35**
Chapter 8

6, 7, 8, 17

represent the

AC

2

2

2, 4

D

56.

is a right triangle

Geometry in 3 Dimensions

b

n

– 2

(42.

5

46.

The Hubble Space Telescope

P

n

– 1

n

P

n

+ 1

n

= 4

1

1

n

Astronomy

The Hubble Space Telescope

r

x

10

rk

5

17.9 cm

x

600 km

6370 km

not to scale

The figures below are drawn on centimeter grid paper. Find the perimeter of each shaded figure to the nearest tenth.

49. Astronomy The Hubble Space Telescope is orbiting Earth 600 km above Earth’s surface. Earth’s radius is about 6370 km. Use the Pythagorean Theorem to find the distance x from the telescope to Earth’s horizon. Round your answer to the nearest ten kilometers.

2830 km

48. Prove the Pythagorean Theorem.

\[ \text{Given: } \triangle ABC \text{ is a right triangle} \]

Prove: \[ a^2 + b^2 = c^2 \]

(Hint: Begin with proportions suggested by Theorem 7-3 or its corollaries.)

53a. Answers may vary. Sample: \( n = 6; 12, 35, 37 \)

53c. \( (2n)^2 + (n^2 - 1)^2 \)

\[ = 4n^2 + n^4 - 2n^2 + 1 \]

\[ = n^4 + 2n^2 + 1 \]

\[ = (n^2 + 1)^2 \]

53a. Find integers \( j \) and \( k \) so that (a) the two given integers \( j \) and \( k \) represent the lengths of the sides of an acute triangle and (b) the two given integers \( j \) and \( k \) represent the lengths of the sides of an obtuse triangle.

40 – 47. Answers may vary. Samples are given.

40. 4, 5; 6, 7

41. 2, 4; 5

42. 6, 9; 8, 11

43. 5, 10; 11, 12

44. 6, 7; 8, 10

45. 9, 12; 14, 16

46. 8, 17; 18, 19

47. 9, 40; 39, 42

53b. Verify that your answers to part (a) form a Pythagorean triple.

53c. Show that, in general, \( (2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2 \) for any \( n \).

54. Geometry in 3 Dimensions The box at the right is a rectangular solid.

a. Use \( \triangle ABC \) to find the length \( d_1 \) of the diagonal of the base. 5 in.

b. Use \( \triangle ABD \) to find the length \( d_2 \) of the diagonal of the box. \( \sqrt{29} \)

c. You can generalize the steps in parts (a) and (b).

Use the facts that \( AC^2 + BC^2 = d_1^2 \) and \( d_2 = \sqrt{BD^2 + AC^2 + BC^2} \)

\[ d_1^2 + BD^2 = d_2^2 \]

Write a one-step formula to find \( d_2 \).

d. Use the formula you wrote to find the length of the longest fishing pole you can pack in a box with dimensions 18 in., 24 in., and 16 in. \( \sqrt{34} \)

55. \( P(0, 0, 0); Q(1, 2, 3) \)

56. \( P(0, 0, 0); Q(-3, 4, -6) \)

57. \( P(-1, 3, 5); Q(2, 1, 7) \)

\( \sqrt{14} \)

\( \sqrt{61} \)

\( \sqrt{17} \)

Real-World Connection

Research by Edwin Hubble (1889–1953), here guiding a telescope in 1923, led to the Big Bang Theory of the formation of the universe.

Alternative Assessment

Have students use the Pythagorean Theorem to find the length of the diagonal of their notebook paper and explain in writing how the Pythagorean Theorem was used. Then have them measure the diagonal to confirm the length found using the Pythagorean Theorem.
58. Use the plan and write a proof of Theorem 8-2, the Converse of the Pythagorean Theorem.

**Given:** ΔABC with sides of length a, b, and c where \(a^2 + b^2 = c^2\)

**Prove:** ΔABC is a right triangle.

**Plan:** Draw a right triangle (not ΔABC) with legs of lengths a and b. Label the hypotenuse x. By the Pythagorean Theorem, \(a^2 + b^2 = x^2\). Use substitution to compare the lengths of the sides of your triangle and ΔABC. Then prove the triangles congruent.  

**See margin.**

59. The lengths of the legs of a right triangle are 17 m and 20 m. To the nearest tenth of a meter, what is the length of the hypotenuse? 26.2

60. The hypotenuse of a right triangle is 34 ft. One leg is 16 ft. Find the length of the other leg in feet. 30

61. What whole number forms a Pythagorean triple with 40 and 41? 9

62. The two shorter sides of an obtuse triangle are 20 and 30. What is the least whole number length possible for the third side? 37

63. Each leg of an isosceles right triangle has measure 10 cm. To the nearest tenth of a centimeter, what is the length of the hypotenuse? 14.1

64. The legs of a right triangle have lengths 3 and 4. What is the length, to the nearest tenth, of the altitude to the hypotenuse? 2.7

**Test Prep**

A sheet of blank grids is available in the Test-Taking Strategies with Transparencies booklet. Give this sheet to students for practice with filling in the grids.

**Resources**

For additional practice with a variety of test item formats:
- Standardized Test Prep, p. 465
- Test-Taking Strategies, p. 460
- Test-Taking Strategies with Transparencies

58. Draw right ΔFDE with legs \(DE\) of length a and \(EF\) of length b, and hyp. of length x. Then \(a^2 + b^2 = x^2\) by the Pythagorean Thm. We are given ΔABC with sides of length a, b, c and \(a^2 + b^2 = c^2\). By subst., \(c^2 = x^2\), so \(c = x\). Since all side lengths of ΔABC and ΔFDE are the same, ΔABC ≅ ΔFDE by SSS. \(\angle C = \angle E\) by CPCTC, so \(m\angle C = 90\). Therefore, ΔABC is a right Δ.