8-3

1. Plan

Objectives
1 To use tangent ratios to determine side lengths in triangles

Examples
1 Writing Tangent Ratios
2 Real-World Connection
3 Using the Inverse of Tangent

Professional Development

Math Background

The Greek mathematicians Hipparchus of Nicaea and Claudius Ptolemy created the discipline of trigonometry more than 2000 years ago, primarily to study astronomy. Subsequent Indian and Arabic mathematicians also developed trigonometry.

More Math Background: p. 414C

Lesson Planning and Resources

See p. 414E for a list of the resources that support this lesson.

PowerPoint

Bell Ringer Practice

Check Skills You'll Need
For intervention, direct students to:

Solving Proportions
Lesson 7-1: Example 3
Extra Skills, Word Problems, Proof Practice, Ch. 7

Using 45°-45°-90° Triangles
Lesson 8-2: Examples 1 and 2
Extra Skills, Word Problems, Proof Practice, Ch. 8

Using 30°-60°-90° Triangles
Lesson 8-2: Example 4
Extra Skills, Word Problems, Proof Practice, Ch. 8

8-3

The Tangent Ratio

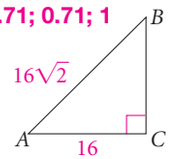
What You'll Learn
- To use tangent ratios to determine side lengths in triangles

. . . And Why
To use the tangent ratio to estimate distance to a distant object, as in Example 2

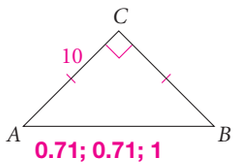
Check Skills You'll Need GO for Help Lessons 7-1 and 8-2

Find the ratios $\frac{BC}{AB}$, $\frac{AC}{AB}$, and $\frac{BC}{AC}$. Round answers to the nearest hundredth.

1. 0.71; 0.71; 1

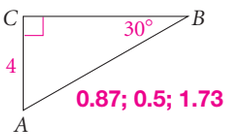


2.



0.71; 0.71; 1

3.



0.87; 0.5; 1.73

Algebra Solve each proportion.

4. $\frac{x}{3} = \frac{4}{7}$ $\frac{12}{7}$
5. $\frac{6}{11} = \frac{x}{9}$ $\frac{54}{11}$
6. $\frac{8}{15} = \frac{4}{x}$ $\frac{15}{2}$
7. $\frac{5}{x} = \frac{7}{12}$ $\frac{60}{7}$

New Vocabulary • tangent

1 Using Tangents in Triangles

Activity: Tangent Ratios

Work in groups of three or four.

- Have your group select one angle measure from {10°, 20°, . . . , 80°}. Then have each member of your group draw a right triangle, $\triangle ABC$, where $\angle A$ has the selected measure. Make the triangles different sizes.
- Measure the legs of each $\triangle ABC$ to the nearest millimeter.

Check students' work.

1. Compute the ratio $\frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A}$ and round to two decimal places.
2. Compare the ratios in your group. Make a conjecture. See left.

2. Answers may vary. Sample: For each ∠, the ratio is the same no matter how large or small the △.

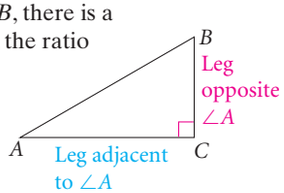
Vocabulary Tip

Trigonometry comes from the Greek words *trigonon* and *metria* meaning "triangle measurement."

For each pair of complementary angles, $\angle A$ and $\angle B$, there is a family of similar right triangles. In each such family, the ratio

$$\frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$

is constant no matter the size of $\triangle ABC$. This trigonometric ratio is called the tangent ratio.



$$\text{tangent of } \angle A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$$

You can abbreviate this equation as $\tan A = \frac{\text{opposite}}{\text{adjacent}}$.

Differentiated Instruction Solutions for All Learners

Special Needs L1
Have students go outside to measure distances indirectly by applying the method in Example 2.
learning style: tactile

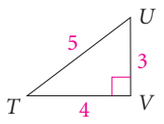
Below Level L2
Have students use all the ratios from the Activity to make a table of tangent values.
learning style: visual

1 EXAMPLE Writing Tangent Ratios

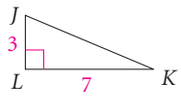
Write the tangent ratios for $\angle T$ and $\angle U$.

$$\tan T = \frac{\text{opposite}}{\text{adjacent}} = \frac{UV}{TV} = \frac{3}{4}$$

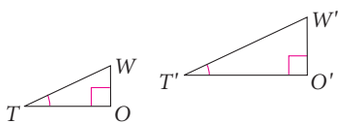
$$\tan U = \frac{\text{opposite}}{\text{adjacent}} = \frac{TV}{UV} = \frac{4}{3}$$



Quick Check 1 **a.** Write the tangent ratios for $\angle K$ and $\angle J$. $\frac{3}{7}; \frac{7}{3}$
b. How is $\tan K$ related to $\tan J$?
They are reciprocals.



As stated on the facing page, the tangent ratio for an acute angle does not depend on leg lengths of a right triangle. To see why this is so, consider the congruent angles, $\angle T$ and $\angle T'$, in the two right triangles shown here.



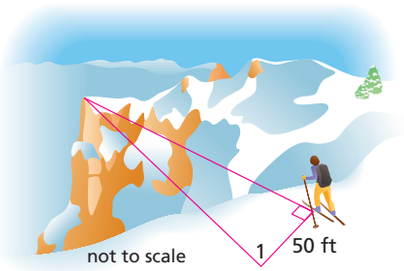
$\triangle TOW \sim T'O'W'$ **AA Similarity Postulate**

$\frac{OW}{TO} = \frac{O'W'}{T'O'}$ **Corresponding sides of ~ triangles are proportional.**

$\tan T = \tan T'$ **Substitute.**

You can use the tangent ratio to measure distances that would be difficult to measure directly.

2 EXAMPLE Real-World Connection



Cross-Country Skiing Your goal in Bryce Canyon National Park is the distant cliff. About how far away is the cliff?

Step 1 Point your compass at a distinctive feature of the cliff and note the reading.

Step 2 Turn 90° and stride 50 ft in a straight path.

Step 3 Turn and point the compass again at the same feature seen in Step 1. Take a reading.

Suppose in Step 3, you find that $m\angle 1 = 86$. The distance you walked in Step 2 was 50 ft. To find the distance to the cliff use the tangent ratio.

$\tan 86° = \frac{x}{50}$ **Use the tangent ratio.**

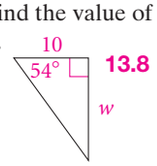
$x = 50(\tan 86°)$ **Solve for *x*.**

50 TAN 86 ENTER 715.03331 **Use a calculator.**

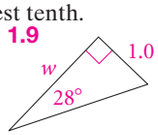
The cliff is about 715 ft away.

Quick Check 2 Find the value of w to the nearest tenth.

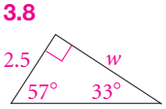
a.



b. 1.9



c. 3.8



2. Teach

Guided Instruction

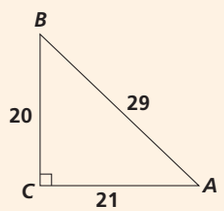
Activity
Because of measurement errors and rounding, students in a group may find that their ratios are close but not equal. If this occurs, ask students to explain why the ratios vary.

Math Tip
Point out that this definition of the tangent of an angle cannot be used to find tan 90° because a right angle has no opposite leg.

Teaching Tip
After students complete Example 1, ask: *What can you conclude about the tangents of complementary angles?* **They are reciprocals.**

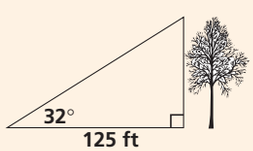
PowerPoint Additional Examples

1 Write the tangent ratios for $\angle A$ and $\angle B$.



$\tan A = \frac{20}{21}; \tan B = \frac{21}{20}$

2 To measure the height of a tree, Alma walked 125 ft from the tree and measured a 32° angle from the ground to the top of the tree. Estimate the height of the tree.



about 78 ft

.

Advanced Learners L4
Have students explain how to find tan 30° and tan 60° without using a calculator, and then confirm the values with a calculator.

learning style: verbal

English Language Learners ELL
Point out that a directional compass is different from the compass used in geometric constructions. Pass around a directional compass and demonstrate its use.

learning style: tactile

Error Prevention!

Students may confuse the angle whose tangent is 5, $\tan^{-1}5$, with $\tan 5^{-1}$. Point out that $\tan 5^{-1} = \tan \frac{1}{5}$, but $\tan^{-1}5$ represents the inverse trigonometric function, or *the angle whose tangent is 5.*

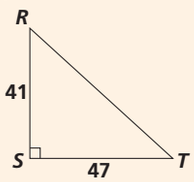
3 EXAMPLE

Students should repeat the example on their own calculators to determine their correct key sequences.

PowerPoint

Additional Examples

3 Find $m\angle R$ to the nearest degree.



$m\angle R \approx 49$

Resources

- Daily Notetaking Guide 8-3 L3
- Daily Notetaking Guide 8-3—Adapted Instruction L1

Closure

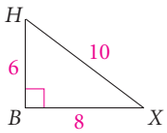
Without using a calculator, find the angle whose tangent equals 1. Explain. **Sample: 45°; by the Converse of the Isosceles Triangle Theorem, a 45°-45°-90° triangle has congruent legs, so $\tan 45° = \frac{1}{1} = 1$.**

If you know leg lengths for a right triangle, you can find the tangent ratio for each acute angle. Conversely, if you know the tangent ratio for an angle, you can use inverse of tangent, $\tan^{-1}$, to find the measure of the angle.

For: Tangent Activity
Use: Interactive Textbook, 8-3

3 EXAMPLE Using the Inverse of Tangent

The lengths of the sides of $\triangle BHX$ are given. Find $m\angle X$ to the nearest degree.



$\tan X = \frac{6}{8} = 0.75$ **Find the tangent ratio.**

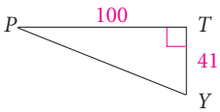
$m\angle X = \tan^{-1}(0.75)$ **Use the inverse of tangent.**

TAN^{-1} 0.75 ENTER 36.869898 **Use a calculator.**

So $m\angle X \approx 37$.

Quick Check

3 Find $m\angle Y$ to the nearest degree. **68**



EXERCISES

For more exercises, see *Extra Skill, Word Problem, and Proof Practice.*

Practice and Problem Solving

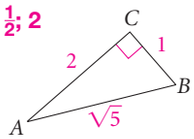
A Practice by Example

Example 1 (page 433)

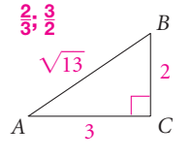
GO for Help

Write the tangent ratios for $\angle A$ and $\angle B$.

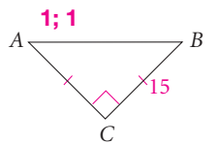
1. $\frac{1}{2}$; 2



2. $\frac{2}{3}$; $\frac{3}{2}$



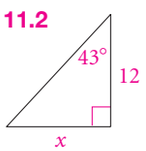
3. 1; 1



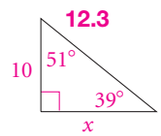
Example 2 (page 433)

Find the value of x to the nearest tenth.

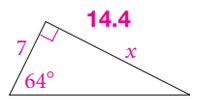
4. 11.2



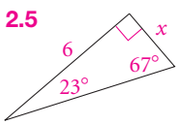
5. 12.3



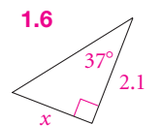
6. 14.4



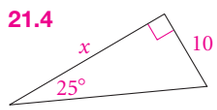
7. 2.5



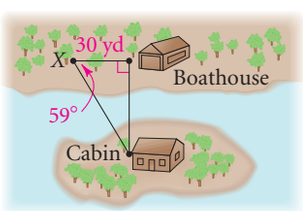
8. 1.6



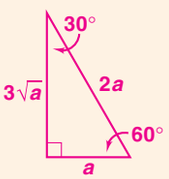
9. 21.4



10. **Surveying** To find the distance from the boathouse on shore to the cabin on the island, a surveyor measures from the boathouse to point X as shown. He then finds $m\angle X$ with an instrument called a transit. Use the surveyor's measurements to find the distance from the boathouse to the cabin. **about 50 yd**



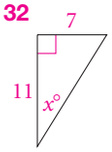
24. **Consider a 30-60-90 △. Let the length of the shorter side be a. Then the length of the longer side, opposite the 60°∠, is $a\sqrt{3}$. Thus, $\tan 60° = \frac{a\sqrt{3}}{a} = \sqrt{3}$.**



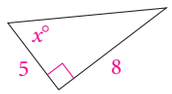
Example 3 (page 434)

Find the value of x to the nearest degree.

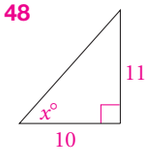
11. 32



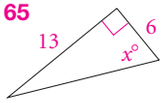
12. 58



13. 48

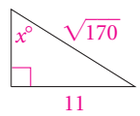


14. 65



15. 63

16. 58



Find each missing value to the nearest tenth.

17. $\tan \blacksquare° = 3.5$ 74.1

18. $\tan 34° = \frac{\blacksquare}{20}$ 13.5

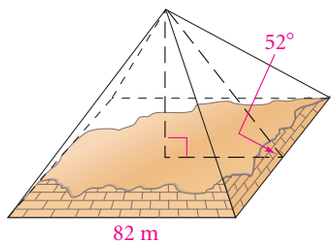
19. $\tan 2° = \frac{4}{\blacksquare}$ 114.5

20. $\tan \blacksquare° = 90$ 89.4

B Apply Your Skills

21. The lengths of the diagonals of a rhombus are 2 in. and 5 in. Find the measures of the angles of the rhombus to the nearest degree. 44 and 136

22. Pyramids All but two of the pyramids built by the ancient Egyptians have faces inclined at 52° angles. Suppose an archaeologist discovers the ruins of a pyramid. Most of the pyramid has eroded, but she is able to determine that the length of a side of the square base is 82 m. How tall was the pyramid, assuming its faces were inclined at 52°? Round your answer to the nearest meter. 52 m



23. Multiple Choice The legs of a right triangle have lengths 5 and 12. What is the tangent of the angle opposite the leg with length 12? D

(A) $\frac{5}{13}$ (B) $\frac{12}{13}$ (C) $\frac{13}{12}$ (D) $\frac{12}{5}$

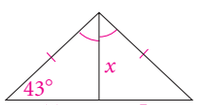
24. Writing Explain why $\tan 60° = \sqrt{3}$. Include a diagram with your explanation.

25. Explain why $\tan^{-1} \frac{\sqrt{2}}{\sqrt{2}} = 45°$. 24–25. See margin.

26. GPS A rectangle is 80 cm long and 20 cm wide. To the nearest degree, find the measures of the angles formed by the diagonals at the center of the rectangle. 152 and 28

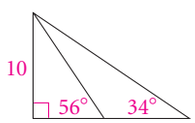
Find the value of w, then x. Round lengths of segments to the nearest tenth. Round angle measures to the nearest degree.

27.



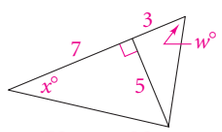
$w = 5; x = 4.7$

28.



$w = 6.7; x = 8.1$

29.



$w = 59; x = 36$

30. a. Coordinate Geometry Complete the table of values at the right. Give table entries to the nearest tenth.

x	$\tan x°$
5	■
10	■
⋮	⋮
85	■

b. Plot the points $(x, \tan x°)$ on the coordinate plane. Connect the points with a smooth curve. See margin.

c. What happens to the tangent ratio as the angle measure x approaches 0? Approaches 90?

d. Use your graph to estimate each value. a, c–d. See left.

$\tan \blacksquare° = 7$ $\tan 68° = \blacksquare$ $\tan \blacksquare° = 3.5$

30a. 0, 1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 1; 1.2; 1.4; 1.7; 2.1; 2.7; 3.7; 5.7; 11.4

30c. approaches 0; increases to infinity

30d. Answers may vary. Samples: 82; 2.5; 74

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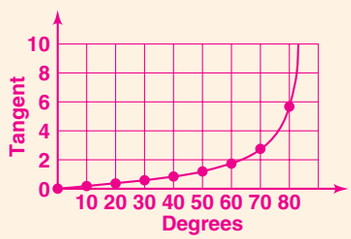
25. $\frac{\sqrt{2}}{\sqrt{2}} = 1$, so we have to show $\tan^{-1} 1 = 45°$. This is equivalent to showing $1 = \tan 45°$.

Consider a 45-45-90 △. Let the lengths of the shorter sides be a. Thus, $\tan 45° = \frac{a}{a} = 1$.

30. b.



3. Practice

Assignment Guide

1 A B	1-45
C Challenge	46-58
Test Prep	59-65
Mixed Review	66-70

Homework Quick Check

To check students' understanding of key skills and concepts, go over Exercises 10, 14, 24, 26, 32.

Technology Tip

Make sure that students set their calculators in degree mode.

Alternative Method

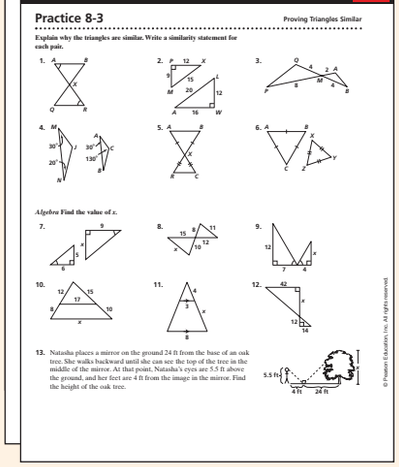
Exercise 9 Point out that students can solve a simpler equation by using the ratio $\tan(90 - 25)° = \frac{x}{10}$.

Exercises 15, 16 Students need to apply the Pythagorean Theorem before they can find the value of x.

Exercise 23 Students should sketch and label the triangle, and be careful to find the tangent ratio asked for.

Differentiated Instruction Resources

GPS Guided Problem Solving L3
Enrichment L4
Reteaching L2
Adapted Practice L1
Practice L3


Practice 8-3 Proving Triangles Similar

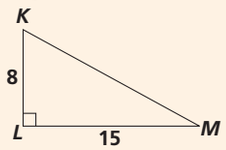
Explain why the triangles are similar. Write a similarity statement for each pair.

4. Assess & Reteach

PowerPoint

Lesson Quiz

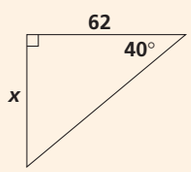
Use the figure for Exercises 1–3.



1. Write the tangent ratio for $\angle K$. $\frac{15}{8}$
2. Write the tangent ratio for $\angle M$. $\frac{8}{15}$
3. Find $m\angle M$ to the nearest degree. 28

Find x to the nearest whole number.

4.



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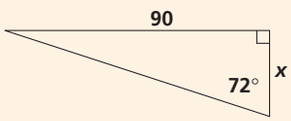
5.



29

Alternative Assessment

Have students draw a right triangle, measure one acute angle and the leg adjacent to it, and then use the tangent function to estimate the length of the other leg. Have students draw another right triangle, measure each leg, and use the inverse tangent function to estimate both acute angles of the triangle.

Real-World Connection

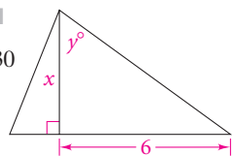
The world's steepest railway is the Katoomba Scenic Railway in Australia's Blue Mountains.

Engineering **The grade of a road or a railway road bed is the ratio $\frac{\text{rise}}{\text{run}}$, usually expressed as a percent. For example, a railway with a grade of 5% rises 5 ft for every 100 ft of horizontal distance.**

31. The Katoomba Railway, pictured at left, has a grade of 122%. What angle does its roadbed make with the horizontal? about 51°
32. The Johnstown, Pennsylvania, inclined railway was built as a "lifesaver" after the Johnstown flood of 1889. It has a 987-ft run at a 71% grade. How high does this railway lift its passengers? about 701 ft
33. The Fenelon Place Elevator railway in Dubuque, Iowa, lifts passengers 189 ft to the top of a bluff. It has an 83% grade. How long is this railway? about 296 ft
34. The Duquesne Incline Plane Company's roadway in Pittsburgh, Pennsylvania, climbs Mt. Washington, located above the mouth of the Monongahela River. It reaches a height of 400 ft with a 793-ft incline. What is its grade? about 58.4%

Find the missing value to the nearest tenth.

35. $x = 2, y =$ ■ 71.6
36. $x = 2\sqrt{3}, y =$ ■ 60.0
37. $x = 6, y =$ ■ 45.0
38. $x = 6\sqrt{3}, y =$ ■ 30.0
39. $x =$ ■, $y = 15$ 22.4
40. $x =$ ■, $y = 30$ 10.4
41. $x =$ ■, $y = 45$ 6.0
42. $x =$ ■, $y = 60$ 3.5
43. $x =$ ■, $y = 75$ 1.6



44. a. **Critical Thinking** Does $\tan A + \tan B = \tan (A + B)$ when $A + B < 90$? Explain. a–b. See margin.
    b. **Reasoning** Does $\tan A - \tan B = \tan (A - B)$ when $A - B > 0$? Use part (a) and indirect reasoning to explain.

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45. **Graphing Calculator** Use the **TABLE** feature of your graphing calculator to study tan X as X gets close to 90. In the Y= screen, enter Y1 = tan X.
    a. Use the **TBLSET** feature so that X starts at 80 and changes by 1. Access the **TABLE**. From the table, what is tan X for X = 89? 57.290
    b. Perform a "numerical zoom in." Use the **TBLSET** feature, so that X starts with 89 and changes by 0.1. What is tan X for X = 89.9? 572.96 c. See margin.
    c. Continue to numerically zoom in on values close to 90. What is the greatest value you can get for tan X on your calculator? How close is X to 90?
    d. **Writing** Use right triangles to explain the behavior of tan X found above. See margin.

Challenge

46. **Graphing Calculator** Use the **TABLE** and graphing features of your graphing calculator to study the product tan X · tan (90 − X). In the Y= screen, enter Y1 = tan X · tan (90 − X).
    a. Use the **TBLSET** feature so that X starts at 1 and changes by 1. Access the **TABLE**. What do you notice? Every Y_1 value = 1.
    b. Press GRAPH. What do you notice? The graph is that of $Y_1 = 1$.
    Proof c. Make a conjecture about tan X · tan (90 − X) based on parts (a) and (b). Write a paragraph proof of your conjecture. See back of book.

Use the given information and $\tan^{-1}$ to find $m\angle A$ to the nearest whole number.

47. $\tan 2A = 9.5144$ 42
48. $\tan \frac{A}{3} = 0.4663$ 75
49. $(\tan 5A)^2 = 0.3333$ 6
50. $\frac{\tan A}{1 + \tan A} = 0.5437$ 50

Problem Solving Hint

Think of $\tan^{-1}(x)$ as "the angle whose tangent is x."

Simplify each expression.

51. $\tan (\tan^{-1} x)$ x
52. $\tan^{-1} (\tan X)$ $m\angle X$

44. a. No; Answers may vary. Sample: tan 45° + tan 30° ≈ 1 + 0.6 = 1.6, but tan(45 + 30)° = tan 75° ≈ 3.7
    b. No; assume tan $A°$ − tan $B°$ = tan$(A - B)°$, or tan $A°$ = tan $B°$ + tan $(A - B)°$. Let $A = B + C$, so by subst., tan$(B + C)°$ = tan $B°$ + tan $C°$. This is false by part (a).

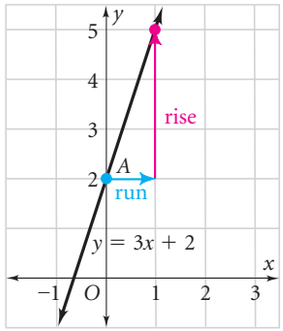
45. c. Answers may vary. Sample: tan $X°$ ≈ 572,958 for $X = 89.9999$
    d. Answers may vary. Sample: In a rt. △, as an acute ∠ approaches 90°, the opp. side gets longer.

Coordinate Geometry **You can use the slope of a line to find the measure of the acute angle that the line forms with any horizontal line.**

$$\text{slope} = \frac{\text{rise}}{\text{run}} = 3$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = 3$$

$$m\angle A = \tan^{-1}(3) \approx 71.6$$



To the nearest tenth, find the measure of the acute angle that the line forms with a horizontal line.

53. $y = \frac{1}{2}x + 6$ **26.6**

54. $y = 6x - 1$ **80.5**

55. $y = 5x - 7$ **78.7**

56. $y = \frac{4}{3}x - 1$ **53.1**

57. $3x - 4y = 8$ **36.9**

58. $-2x + 3y = 6$ **33.7**

Test Prep

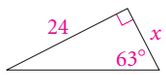
Gridded Response

59. What is tan 84° to the nearest tenth? **9.5**

60. What is the whole number value of $\tan^{-1}\sqrt{3}$? **60**

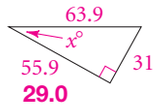
In Exercises 61–64 what is the value of x to the nearest tenth?

61.



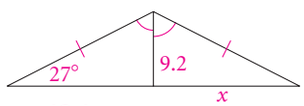
12.2

62.



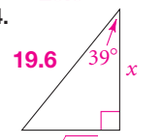
29.0

63.



18.1

64.



65. The tangent of an angle is 7.5. What is the measure of the angle to the nearest tenth? **82.4**

Mixed Review

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Lesson 8-2

66. A diagonal of a square is 10 units. Find the length of a side of the square. Leave your answer in simplest radical form. **$5\sqrt{2}$ units**

Lesson 8-1

The lengths of the sides of a triangle are given. Classify each triangle as *acute*, *right*, or *obtuse*.

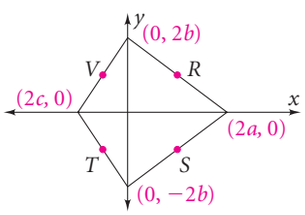
67. 5, 8, 4 **obtuse**

68. 15, 15, 20 **acute**

69. 0.5, 1.2, 1.3 **right**

Lesson 6-7

70. For the kite pictured at the right, give the coordinates of the midpoints of its sides.
***R*(*a*, *b*); *S*(*a*, −*b*); *T*(*c*, −*b*); *V*(*c*, *b*)**



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Test Prep

Resources

For additional practice with a variety of test item formats:

- Standardized Test Prep, p. 465
- Test-Taking Strategies, p. 460
- Test-Taking Strategies with Transparencies